STABILISED FINITE ELEMENT METHODS FOR NON-SYMMETRIC, NON-COERCIVE AND ILL-POSED PROBLEMS. PART I: ELLIPTIC EQUATIONS

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Abstract. In this paper we propose a new method to stabilise non-symmetric indefinite problems. The idea is to solve a forward and an adjoint problem simultaneously using a suitable stabilised finite element method. Both stabilisation of the element residual and jumps of certain derivatives of the discrete solution over element faces may be used. Under the assumption of well posedness of the partial differential equation and its associated adjoint problem we prove optimal error estimates in H^1 and L^2 norms in an abstract framework. Some examples of problems that are neither symmetric nor coercive, but that enter the abstract framework are given. First we treat indefinite convection-diffusion equations, with non-solenoidal transport velocity and either pure Dirichlet conditions or pure Neumann conditions and then a Cauchy problem for the Helmholtz operator. Some numerical illustrations are given.

1. Introduction. The computation of indefinite elliptic problems often involves certain conditions on the mesh size h for the system to be well-posed and for the derivation of error estimates. The first results on this problem are due to Schatz [23]. Such conditions on the mesh parameter can be avoided if a stabilised finite element method is used. Such methods have been propsed by Bramble et al. [5, 20] or more recently the conitnuous interior penalty (CIP) method for the Helmholtz equation suggested by Wu et al. [25, 24]. The method proposed herein has some common features with both these methods, but appears to have a wider field of applicability. We may treat not only symmetric indefinite problems such as the (real valued) Helmholtz equation, but also nonsymmetric indefinite problems such as convection–diffusion problems with non-solenoidal convection velocity or the Cauchy problem. The latter problem is known to be ill-posed in general [2] and will, in the ill-posed case, mainly be explored numerically herein. For all these cases we show that if the primal and adjoint problems admit a unique solution with sufficient smoothness the proposed algorithm converges with optimal order. The case of hyperbolic problems is treated in the companion paper [6].

The idea of this work is to assume ill-posedness of the discrete form of the PDE and regularise it in the form of an optimisation problem under constraints. Indeed we seek to minimise the size of the stabilisation operator under the constraint of the discrete variational form. The regularisation terms are then chosen from well known stabilised methods respecting certain design criteria given in an abstract analysis. This leads to an extended method where simultanuously both a primal and a dual problem are solved, but where the exact solution of the dual problem is always trivial. The aim is to obtain a method where possible discrete non-uniqueness is alleviated by discrete regularisation with a non-consistency that can be controlled so that optimal convergence for smooth solutions is obtained. The method is also a good candidate for cases where the solution to the continuous problem is non-unique, but that is beyond the scope of the present paper.

In spite of the lack of coercivity for the physical problem, the discrete problem has partial coercivity on the stabilisation operator. A consequence of this is that, depending on the kernel of the stabilisation operator a unique discrete solution may

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often be shown to exist independently of the underlying partial differential equation. This can be helpful when exploring ill-posed problems numerically or when measurement errors in the data, may lead to an ill-posed problem, although the true problem is well posed.

An outline of the paper is as follows, in Section 2 we propose an abstract method and prove that the method will have optimal convergence under certain assumptions on the bilinear form. Then in Section 3 we discuss stabilised methods that satisfy the assumptions of the abstract theory with particular focus on the Galerkin least squares method (GLS) and the continuous interior penalty method (CIP). Three example of applications are given in Section 4, two different non-coercive transport problems in compressible flow fields and one elliptic Cauchy problem. Finally in Section 5 the accuracy and robustness of the proposed method is shown by some computations of solutions to the problems discussed in Section 4. In particular we study the performance of the approach for some different Cauchy problems of varying difficulty.

2. Abstract formulation. Let Ω be a polygonal/polyhedral subset of $\mathbb{R}^d$. For simplicity we will reduce the scope to second order elliptic problems, but the methodology can readily be extended to indefinite elliptic problems of any order, providing the operator has a smoothing property.

We let V, W denote two subspaces of $H^1(\Omega)$. The abstract weak formulation of the continuous problem takes the form: find $u \in V$ such that

$$a(u, v) = (f, v), \quad \forall v \in W \tag{2.1}$$

with formal adjoint: find $z \in W$ such that

$$a(w, z) = (g, w), \quad \forall w \in V. \tag{2.2}$$

The bilinear form $a(\cdot, \cdot) : V \times W \to \mathbb{R}$ is assumed to be elliptic, but neither symmetric nor coercive. We denote the forward problem on strong form $\mathcal{L}u = f$ and the adjoint problem on strong form $\mathcal{L}^* z = g$. Suitable boundary conditions are integrated either in the spaces V, W or in the linear form.

We assume that both these problems are well posed and that the geometry and data are such that the smoothing property holds

$$|u|_{H^2(\Omega)} \leq c_{a,\Omega} \|f\|, \quad |z|_{H^2(\Omega)} \leq c_{a,\Omega} \|g\|. \tag{2.3}$$

We will frequently use the notation $a \lesssim b$ for $a \leq Cb$ with C a constant depending only on the mesh geometry and physical parameters of order one (or irrelevant in the context). We will also use $a \sim b$ for $a \lesssim b$ and $b \lesssim a$. Indexed constants c_{xy} will have similar dependence on the variables xy at each occurence but can differ by a constant factor.

The L^2-scalar product over some $X \subset \mathbb{R}^d$ is denoted $(\cdot, \cdot)_X$ and the associated norm $\| \cdot \|_X$, the subscript is dropped whenever $X = \Omega$. We will also use $\langle \cdot, \cdot \rangle_Y$ to denote the L^2-scalar product over $Y \subset \mathbb{R}^{d-1}$ and $(\cdot, \cdot)_h$ the element-wise L^2-norm with the associated broken norm $\| \cdot \|_h$.

2.1. Finite element discretisation. Let $\{\mathcal{T}_h\}_h$ denote a family of quasi uniform, shape regular triangulations $\mathcal{T}_h := \{K\}$, indexed by the maximum triangle radius h. The set of faces of the triangulation will be denoted by $\mathcal{F}$ and $\mathcal{F}_{int}$ denotes

the subset of interior faces. Now let X_h^k denote the finite element space of continuous, piecewise polynomial fuctions on $\mathcal{T}_h$,

$$X_h^k := \{v_h \in H^1(\Omega) : v_h|_K \in \mathbb{P}_k(K), \quad \forall K \in \mathcal{T}_h\}.$$

Here $\mathbb{P}_k(K)$ denotes the space of polynomials of degree less than or equal to k on a triangle K.

We let π_L denote the standard L^2-projection onto X_h^k and $i_h : C^0(\bar{\Omega}) \mapsto X_h^k$ the standard Lagrange interpolant. Recall that for any function $u \in (V \cup W) \cap H^{k+1}(\Omega)$ there holds

$$\|u - i_h u\| + h\|\nabla(u - i_h u)\| + h^2\|D^2(u - i_h u)\|_h \leq c_i h^{k+1}|u|_{H^{k+1}(\Omega)}. \tag{2.4}$$

and under our assumptions on the mesh, similarly for π_L. We propose the following finite element method for the approximation of (2.1), find $(u_h, z_h) \in V_h \times W_h$ such that

$$\begin{aligned} a_h(u_h, v_h) + s_a(z_h, v_h) &= (f, v_h) \\ a_h(w_h, z_h) - s_p(u_h, w_h) &= -s_p(u, w_h), \end{aligned} \tag{2.5}$$

for all $(v_h, w_h) \in W_h \times V_h$. The spaces V_h and W_h are typically both chosen as X_h^k, possibly with some additional constraint. If both the primal and the adjoint variable are approximated in the same spaces, we use the notation V_h for both. The bilinear form $a_h(\cdot,\cdot)$ is a discrete realisation of $a(\cdot,\cdot)$, typically modified to account for the effect of boundary conditions, since in general $V_h \not\in V$ and $V_h \not\in W$. The penalty operators $s_a(\cdot,\cdot)$ and $s_p(\cdot,\cdot)$ are symmetric stabilisation operators and associated with the adjoint and the primal equation respectively. The rationale of the formulation may be explained in an optimisation framework. Assume that we want to solve the problem, find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h,$$

but that the system matrix corresponding to $a_h(u_h, v_h)$ has zero eigenvalues. The discrete system is ill-posed. This often reflects some poor stability properties of the underlying continuous problem. The idea is to introduce a silection criterion for the solution, in order to ensure discrete uniqueness, measured by some operator $s_p(u_h, v_h)$. This can include both stabilisation (regularisation) terms and the fitting of the computed solution to measurements or some goal function. Interestingly, as we shall see, finite element stabilisation terms are the only regularisation that is needed. The formulation then writes, find $u_h, z_h \in V_h \times W_h$ stationary point of the Lagrangian

$$\text{Ł}(u_h, z_h) := \frac{1}{2}s_p(u_h - u, u_h - u) - \frac{1}{2}s_a(z_h, z_h) - a_h(u_h, z_h) + (f, z_h). \tag{2.6}$$

The saddle point structure of the Lagrangian has been strengthened by the addition of the regularizing term $-\frac{1}{2}s_a(z_h, z_h)$. We may readily verify that

$$\frac{\partial \text{Ł}}{\partial u_h}(w_h) = s_p(u_h - u, w_h) - a_h(w_h, z_h)$$

and

$$\frac{\partial \text{Ł}}{\partial z_h}(v_h) = -a_h(u_h, v_h) - s_a(z_h, v_h) + (f, v_h).$$

It follows that (2.5) corresponds to the optimality conditions of (2.6).

Observe that the second equation of (2.5) is a finite element discretisation of the dual problem (2.2) with data $g = 0$. Hence the solution to approximate is $z = 0$, however z_h will most likely not be zero, since it is perturbed by the stabilisation operator acting on the solution u_h, which in general does not coincide with the stabilisation acting on u.

We will assume that the following strong consistency properties hold. If u is the solution of (2.1) then

$$a_h(u, \varphi) = (\mathcal{L}u, \varphi) \text{ for all } \varphi \in W + W_h \tag{2.7}$$

and if z is the solution of (2.2) then

$$a_h(\phi, z) = (\phi, \mathcal{L}^* z) \text{ for all } \phi \in V + V_h. \tag{2.8}$$

As a consequence the following Galerkin orthogonalities hold

$$a_h(u - u_h, v_h) = s_a(z_h, v_h) \text{ and } a_h(w_h, z - z_h) = s_p(u - u_h, w_h). \tag{2.9}$$

The bilinear forms $s_a(\cdot,\cdot)$, $s_p(\cdot,\cdot)$ are symmetric, positive semi-definite, weakly consistent, stabilisation operators. For simplicity we will always assume that u is sufficiently regular so that $s_p(u, v_h)$ is well defined, i.e. the stabilisation is strongly consistent. The analysis usin weak consistency of the stabilisation only is a straightforward modification. The semi-norms on V_h and W_h associated to the stabilisation is defined by

$$|x_h|_{S_y} := s_y(x_h, x_h)^{\frac{1}{2}}, \quad y = a, p.$$

The rationale of this formulation is that the following partial coercivity is obtained by taking $v_h = z_h$ and $w_h = u_h$,

$$|z_h|_{S_a}^2 + |u_h|_{S_p}^2 = (f, z_h) - s_p(u, u_h). \tag{2.10}$$

We assume that there are interpolation operators $\pi_V : V \to V_h$ and $\pi_W : W \to W_h$, satisfying (2.4) and also the following continuity relations for all $v, w, y \in H^2(\Omega)$ and for all $x_h \in W_h$

$$a_h(v - \pi_V v, x_h) \leq \|v - \pi_V v\|_+ (c_a |x_h|_{S_a} + \epsilon(h)\|x_h\|) \tag{2.11}$$

and

$$a_h(u - u_h, y - \pi_W y) \leq \|y - \pi_W y\|_* (c_a \|u - \pi_V u\|_{\mathcal{L}} + c_a |u_- \pi_V u|_{S_p} + \epsilon(h)\|u - u_h\|) \tag{2.12}$$

where $\|\cdot\|_+$, $\|\cdot\|_*$ and $\|\cdot\|_{\mathcal{L}}$ are semi-norms to be defined, satisfying the approximation estimates

$$\|v - \pi_V v\|_{\mathcal{L}} + \|v - \pi_V v\|_+ + |v - \pi_V v|_{S_p} \leq c_{a,\gamma} h^k |v|_{H^{k+1}(\Omega)}, \quad \forall v \in V \cap H^{k+1}(\Omega), \tag{2.13}$$

$$\|w - \pi_W w\|_* + |w - \pi_W w|_{S_a} \leq c_{a,\gamma} h^k |w|_{H^{k+1}(\Omega)} \quad \forall w \in W \cap H^{k+1}(\Omega), \tag{2.14}$$

and

$$|\pi_W w|_{S_a} \leq c_{a,\gamma} h |w|_{H^2(\Omega)}, \ \forall w \in w \cap H^2(\Omega), \text{ and } |\pi_V v|_{S_p} \leq c_{a,\gamma} h |v|_{H^2(\Omega)} \ \forall v \in V \cap H^2(\Omega), \tag{2.15}$$

where $c_{a,\gamma}$ depends on the form $a(\cdot,\cdot)$ and a stabilisation parameter γ.

2.2. Convergence analysis for the abstract method. We first prove that, assuming that the discrete solution (u_h, z_h) exists, the stabilisation semi-norm of the discrete error is bounded by one term that converges to zero at an optimal rate and another non-essential perturbation that can be made small.

LEMMA 2.1. *Assume that the solution of* (2.5) *exists, that the solution of* (2.1) *is smooth, that the forms of* (2.5) *and the operators* π_V, π_W *are such that* (2.9), (2.11) *and* (2.13) *are satisfied. Then there holds*

$$|\pi_V u - u_h|_{S_p} + |\pi_W z - z_h|_{S_a} \leq c_{a,\gamma,\epsilon} h^k |u|_{H^{k+1}(\Omega)} + \epsilon(h)\|z_h\|,$$

where $c_{a,\gamma,\epsilon} := c_{a,\gamma}(2+c_a^2)^{\frac{1}{2}}$, *with* c_a *and* $c_{a,\gamma}$ *defined by* (2.11) *and* (2.13) *respectively. Similarly, if* $s_p(u, w_h) = 0$, *there holds*

$$|u_h|_{S_p} + |z_h|_{S_a} \leq (c_{a,\gamma} + c_{a,\gamma,\epsilon}) h^k |u|_{H^{k+1}(\Omega)} + \epsilon(h)\|z_h\|.$$

Proof. For the first inequality let $\xi_h = \pi_V u - u_h$ and $\zeta_h = \pi_W z - z_h$. By the definition (2.5) there holds

$$|\xi_h|_{S_p}^2 + |\zeta_h|_{S_a}^2 = s_p(\xi_h, \xi_h) + s_a(\zeta_h, \zeta_h) = a_h(\xi_h, \zeta_h) + s_a(\zeta_h, \zeta_h) - a_h(\xi_h, \zeta_h) + s_p(\xi_h, \xi_h).$$

Using now the Galerkin orthogonality of $a_h(\cdot,\cdot)$, (2.9), we have

$$|\xi_h|_{S_p}^2 + |\zeta_h|_{S_a}^2 = a_h(\pi_V u - u, \zeta_h) + s_a(\pi_W z, \zeta_h) - a_h(\xi_h, \pi_W z - z) + s_p(\pi_V u - u, \xi_h).$$

Observing that $z = \pi_W z = 0$ this reduces to

$$|\xi_h|_{S_p}^2 + |\zeta_h|_{S_a}^2 = a_h(\pi_V u - u, \zeta_h) + s_p(\pi_V u - u, \xi_h).$$

We conclude by applying the continuity (2.11)

$$|\xi_h|_{S_p}^2 + |\zeta_h|_{S_a}^2 \leq \|u - \pi_V u\|_{+}(c_a|\zeta_h|_{S_a} + \epsilon(h)\|z_h\|) + |u - \pi_V u|_{S_p}|\xi_h|_{S_p}$$

followed by

$$\begin{aligned} |\xi_h|_{S_p}^2 + |\zeta_h|_{S_a}^2 &\leq (c_a^2 + 1)\|u - \pi_V u\|_{+}^2 + \epsilon(h)^2\|z_h\|^2 + |u - \pi_V u|_{S_p}^2 \\ &\leq c_{a,\gamma}^2(1 + c_a^2) h^{2k} |u|_{H^{k+1}(\Omega)}^2 + \epsilon(h)^2\|z_h\|^2. \end{aligned}$$

The second result follows by adding and subtracting $\pi_V u$, observing that here $\pi_W z = 0$, applying a triangle inequality and then (2.13) on $|\pi_V u|_{S_p} = |\pi_V u - u|_{S_p}$. ◻

We may now prove the main result which is optimal convergence in the L^2 and the H^1 norms.

THEOREM 2.2. *Assume that* (2.1) *and* (2.2) *are well-posed with exact solutions* u *and* z *satisfying* (2.3). *Assume that the forms of* (2.5) *and the operators* π_V, π_W *are such that* (2.9)–(2.15) *are satisfied and that* h *is so small that*

$$c_{a,\gamma,\Omega,} h\, \epsilon(h) \leq \frac{1}{6}, \tag{2.16}$$

with $c_{a,\gamma,\Omega}$ *depending mainly on the constants of the inequalities* (2.3) *and* (2.13)–(2.15). *Then the discrete solution* u_h, z_h *exists and satsifies*

$$\|u - u_h\| + h\|\nabla(u - u_h)\| + \|z_h\| \leq C_{a,\Omega,\gamma} h^{k+1} |u|_{H^{k+1}(\Omega)}$$

and in particular

$$\|u-u_h\| + h\|\nabla(u-u_h)\| + \|z_h\| \le C_{a,\Omega,\gamma} h^2 \|f\|. \tag{2.17}$$

Proof. Let φ be the solution of (2.2) with $g = u - u_h$ and ψ the solution of (2.1) with $f = z_h$. By (2.3) there holds

$$\|\varphi\|_{H^2(\Omega)} \le c_{a,\Omega}\|u-u_h\| \text{ and } \|\psi\|_{H^2(\Omega)} \le c_{a,\Omega}\|z_h\|.$$

By definition of the primal and dual problems and by (2.7), (2.8), (2.9), (2.11) and (2.12) there holds,

$$\begin{aligned}\|u-u_h\|^2 + \|z_h\|^2 &= (u-u_h, \mathcal{L}^*\varphi) + (\mathcal{L}\psi, z_h) = a_h(u-u_h,\varphi) + a_h(\psi, z_h)\\ &= a_h(u-u_h, \varphi - \pi_W\varphi) + s_a(z_h, \pi_W\varphi)\\ &\quad + a_h(\psi - \pi_V\psi, z_h) - s_p(u-u_h, \pi_V\psi)\\ &\le (c_a\|u-\pi_V u\|_{\mathcal{L}} + c_a|u_h - \pi_V u|_{S_p} + \epsilon(h)\|u-u_h\|)\|\varphi - \pi_W\varphi\|_*\\ &\quad + (c_a|z_h|_{S_a} + \epsilon(h)\|z_h\|)\|\psi - \pi_V\psi\|_+\\ &\quad + |z_h|_{S_a}|\pi_W\varphi|_{S_a} + |u_h - u|_{S_p}|\pi_V\psi|_{S_p}.\end{aligned}$$

First we observe that by (2.13), (2.14) and (2.3)

$$\begin{aligned}&\epsilon(h)\|u-u_h\|\|\varphi - \pi_W\varphi\|_* + \epsilon(h)\|z_h\|\|\psi-\pi_V\psi\|_+\\ &\qquad\qquad \le c_{a,\gamma,\Omega} h(\epsilon(h)\|u-u_h\|^2 + \epsilon(h)\|z_h\|^2).\end{aligned}$$

Then by Lemma 2.1 and approximation we have

$$\begin{aligned}&|z_h|_{S_a}|\pi_W\varphi|_{S_a} + |u_h-u|_{S_p}|\pi_V\psi|_{S_p}\\ &\qquad \le ((c_{a,\gamma} + c_{a,\gamma,\epsilon})h^k|u|_{H^{k+1}(\Omega)} + \epsilon(h)\|z_h\|)c_{a,\gamma,\Omega}h(\|u-u_h\| + \|z_h\|).\end{aligned}$$

Using the two previous bounds and a arithmetic-geometric inequality we have

$$\begin{aligned}&(1 - 3c_{a,\gamma,\Omega}h\epsilon(h))(\|u-u_h\|^2 + \|z_h\|^2)\\ &\qquad\qquad \le C_{a,\gamma}h^{k+1}|u|_{H^{k+1}(\Omega)}(|\varphi|_{H^2(\Omega)} + |\psi|_{H^2(\Omega)}).\end{aligned}$$

Using equation (2.3), the result for the L^2-norm follows provided h satisfies (2.16). The result for the H^1-norm follows using a global inverse inequality on the discrete error and then the L^2-norm error estimate.

$$\|\nabla(u-u_h)\| \le \|\nabla(u-\pi_V u)\| + \|\nabla(\pi_V u - u_h)\| \lesssim h^k|u|_{H^{k+1}(\Omega)} + h^{-1}\|\pi_V u - u_h\|.$$

The existence of a unique solution to (2.5) is a consequence of (2.17). Well-posedness of (2.1) means that $f = 0$ implies $u = 0$, but then by (2.17) $u_h = z_h = 0$, which shows that the matrix is invertible. ◻

The optimal convergence of the stabilisation terms follows.

Corollary 2.3. *Under the assumptions of Lemma 2.1 and Theorem 2.2 there holds*

$$|\pi_V u - u_h|_{S_p} + |\pi_W z - z_h|_{S_a} \lesssim c_{s,\epsilon}h^k|u|_{H^{k+1}(\Omega)} + O(h^{k+1}).$$

Proof. Immediate consequence of Lemma 2.1 and Theorem 2.2. ◻

Remark 1. The need to control a low order contribution of the dual solution usually comes from oscillation of data. Either in the form of stabilisation terms that do not account for oscillation within the element or error in the numerical quadrature. In case a Gårdings inequality holds for (2.5) and $s_a(\cdot,\cdot) \equiv s_p(\cdot,\cdot)$ the H^1-error can be recovered without using inverse inequalities as stated below.

COROLLARY 2.4. *Assume that for the bilinear form $a(\cdot,\cdot)$ there exists $\lambda \in \mathbb{R}$ such that*

$$\|\nabla v_h\|^2 - \lambda \|v_h\|^2 \lesssim a_h(v_h, v_h) + s_p(v_h, v_h)$$

and that $s_a(\cdot,\cdot) \equiv s_p(\cdot,\cdot)$. Then

$$\|\nabla(u - u_h)\| \lesssim h^k |u|_{H^{k+1}(\Omega)}.$$

Proof. Similar to proof of Lemma 2.1 and therefore only sketched. Let $\xi_h := \pi_V u - u_h$. It follows by Gårdings inequality that

$$\|\nabla \xi_h\|^2 \lesssim a_h(\xi_h, \xi_h) + \lambda \|\xi_h\|^2 + s_p(\xi_h, \xi_h).$$

Using Galerkin orthogonality we have

$$a_h(\xi_h, \xi_h) = a_h(\pi_V u - u, \xi_h) + s_a(z_h, \xi_h)$$

and the rest follows as in Lemma 2.1 by (2.11), (2.13) and using the known convergences of Lemma 2.1 and Theorem 2.2. ◻

3. Stabilisation methods. To fix the ideas let $\mathcal{L}$ be a second order elliptic operator on conservation form,

$$\mathcal{L}u := -\mu \Delta u + \nabla \cdot (\beta u) + cu. \tag{3.1}$$

Here $\mu \in \mathbb{R}^+$, $\beta \in [C^2(\Omega)]^2$ is a non-solenoidal velocity vectorfiel and $c \in C^1(\Omega)$. A necessary condition for the continuities (2.11) and (2.12) are that the the stabilisation operators satisfy

$$\inf_{w_h \in V_h} \sum_K \|h(\mathcal{L}u_h - w_h)\|_K^2 + \|h^{\frac{1}{2}} \mu^{\frac{1}{2}} [\![\nabla u_h \cdot n_F]\!]\|_{\mathcal{F}_{int}}^2 \lesssim s_p(u_h - u, u_h - u) \tag{3.2}$$

$$\inf_{w_h \in V_h} \sum_K \|h(\mathcal{L}^* z_h - w_h)\|_K^2 + \|h^{\frac{1}{2}} \mu^{\frac{1}{2}} [\![\nabla z_h \cdot n_F]\!]\|_{\mathcal{F}_{int}}^2 \lesssim s_a(z_h, z_h) \tag{3.3}$$

at least up to a non-essential low order perturbation. Note that taking $w_h = f$ in the first term in the left hand side results in a least squares term on the residual over the element. It follows that the stabilisation relies on two mechanisms: L^2-control of the element residual and L^2-control of the gradient jump over element edges. If higher order differential equations are considered, jumps of higher derivatives must be added. The design criterion (3.2)-(3.3) makes it straightforward to adapt the analysis below to a range of stabilisation methods, such as Galerkin least squares, orthogonal subscales, continuous interior penalty or discontinuous Galerkin methods. In all cases however the jumps of the gradient must be penalised, or an equivalent stabilisation

operator introduced. It therefore seems natural to consider two stabilisations in more detail, first the GLS-stabilisation combined with gradient penalty and then a CIP-stabilisation purely based on penalty on jumps of derivatives of the approximate solution. We introduce the stabilisation operators

$$s_p(u_h, v_h) := s_{p,GLS}(u_h, v_h) + s_{cip}(u_h, v_h) \tag{3.4}$$

and

$$s_a(z_h, w_h) := s_{a,GLS}(z_h, w_h) + s_{cip}(u_h, v_h) \tag{3.5}$$

where

$$s_{p,GLS}(u_h, v_h) := (\gamma_{GLS} h^2 \mathcal{L} u_h, \mathcal{L} v_h)_h,$$

$$s_{a,GLS}(z_h, w_h) := (\gamma_{GLS} h^2 \mathcal{L}^* z_h, \mathcal{L}^* w_h)_h$$

and

$$s_{cip}(u_h, v_h) := \sum_{F \in \mathcal{F}_{int}} \int_F (h_F \gamma_{1,F} [\![\nabla u_h]\!] \cdot [\![\nabla v_h]\!] + h_F^3 \gamma_{2,F} [\![\Delta u_h]\!][\![\Delta v_h]\!]) \, \mathrm{d}x. \tag{3.6}$$

Here $[\![\nabla u_h]\!]|_F$ and $[\![\Delta u_h]\!]|_F$ denotes the jump of the gradient and the Laplacian over the face F. Note that for smooth u, $s_p(u, v_h) = s_{p,GLS}(u, v_h) = (f, \gamma_{GLS} h^2 L v_h)_h$ showing the Petrov-Galerkin character of the GLS-method. The abstract analysis typically holds for the parameter choices $\gamma_{GLS} > 0$, $\gamma_{1,F} > 0$, $\gamma_{2,F} = 0$ or $\gamma_{GLS} = 0$, $\gamma_{1,F} > 0$, $\gamma_{2,F} > 0$. Note that the matrix stencil for finite element methods remains the same for both approaches, and therefore the CIP-method seems more appealing in this context. Eliminating the Galerkin least squares term also reduce the computational effort since the same stabilisation is used for the primal and adjoint solution. If on the other hand a C^1-continuous approximation space is used, the jumps of the gradients may be omitted and the GLS stabilisation might prove competitive, since integrations on the faces may then be avoided.

3.1. Galerkin-least-squares stabilisation. The GLS-method is one of the most popular stabilised methods. To fix the ideas we will assume that the problems (2.1) and (2.2) are subject to homogeneous Dirichlet conditions and well-posed, with $f \in L^2(\Omega)$. For the readers convenience we detail the Lagrangian (2.6) in this particular case

$$\begin{aligned} \mathrm{L}(u_h, z_h) := \frac{1}{2} \|\tau^{\frac{1}{2}} (\mathcal{L} u_h - f)\|_h^2 + \frac{1}{2} s_{cip}(u_h, u_h) \\ - \frac{1}{2} \|\tau^{\frac{1}{2}} \mathcal{L}^* z_h\|_h^2 - \frac{1}{2} s_{cip}(z_h, z_h) - a(u_h, z_h) + (f, z_h). \end{aligned} \tag{3.7}$$

The optimality conditions then write, find $u_h, z_h \in V_h \cap H_0^1(\Omega)$

$$\begin{aligned} a(u_h, v_h) + s_a(z_h, v_h) &= (f, v_h) \\ a(w_h, z_h) - s_p(u_h, w_h) &= -s_p(u, w_h) = -(f, \tau \mathcal{L} w_h)_h \end{aligned} \tag{3.8}$$

for all $v_h, w_h \in V_h \cap H_0^1(\Omega)$. Here $\gamma_{GLS} h^2 := \tau > 0$ and $\gamma_{1,F} \sim \mu$, $\gamma_{2,F} = 0$. We assume that the physical parameters are all order unity for simplicity. Observe the

nonstandard structure of the stabilisation terms and that the formulation is consistent for u the exact solution of (2.1) and $z = 0$. We will now prove that the assumptions of Proposition 2.1 and Theorem 2.2 are satisfied for the formulation (3.8).

We define the following semi-norms

$$\|v\|_+ := \|v\|_* := \|\tau^{-\frac{1}{2}} v\| + \left(\sum_{F \in \mathcal{F}_{int}} \|\mu^{\frac{1}{2}} h^{-\frac{1}{2}} v\|_F^2 \right)^{\frac{1}{2}}, \tag{3.9}$$

$$\|v\|_{\mathcal{L}} := \|\tau^{\frac{1}{2}} \mathcal{L} v\|_h + \|h^{\frac{1}{2}} \mu^{\frac{1}{2}} [\![\nabla u \cdot n_F]\!]\|_{\mathcal{F}_{int}}$$

and

$$|x|_{S_p} := \|\tau^{\frac{1}{2}} \mathcal{L} x\|_h + s_{cip}(x,x)^{\frac{1}{2}} \text{ and } |x|_{S_a} := \|\tau^{\frac{1}{2}} \mathcal{L}^* x\|_h + s_{cip}(x,x)^{\frac{1}{2}},$$

defined for $x \in H^2(\Omega) + V_h$. Let π_V and π_W be defined by the Lagrange interpolator i_h (or any other H^2-stable interpolation operator that satisfies boundary conditions), and note that by (2.4) we readily deduce the following approximation results for smooth enough functions u

$$\|u - \pi_V u\|_+ + \|u - \pi_V u\|_{\mathcal{L}} + |u - \pi_V u|_{S_p} + \|u - \pi_W u\|_* + |u - \pi_W u|_{S_a} \leq c_{a,\gamma} h^k |u|_{H^{k+1}(\Omega)}$$

and, by H^2-stability of the interpolation operator,

$$|\pi_V v|_{S_p} \leq c_{\gamma,a} h \|v\|_{H^2(\Omega)}, \quad |\pi_W w|_{S_a} \leq c_{\gamma,a} h \|w\|_{H^2(\Omega)}, \quad \forall v, w \in H^2(\Omega).$$

This shows that (2.13) and (2.14) holds. It then only remains to show the continuities (2.11) and (2.12). First we show the inequality (2.11) For the second order elliptic problem we note that after an integration by parts and Cauchy-Schwarz inequality,

$$\begin{aligned} a_h(v - \pi_V v, x_h) &= \sum_{F \in \mathcal{F}_{int}} \langle u - \pi_V u, [\![\mu \nabla x_h \cdot n_F]\!] \rangle_F + \sum_K (u - \pi_V u, \mathcal{L}^* x_h)_K \\ &\leq \|u - \pi_V u\|_+ |x_h|_{S_a}. \end{aligned}$$

Similarly, to prove (2.12) we integrate by parts in the opposite direction in the second order operator and obtain

$$\begin{aligned} a_h(u - u_h, y - \pi_W y) = \sum_{K \in \mathcal{T}_h} &(\mathcal{L}(u - u_h), y - \pi_W y)_K \\ &+ \sum_{F \in \mathcal{F}_{int}} \langle [\![\mu \nabla u_h \cdot n_F]\!], y - \pi_W y \rangle_F \\ &\|y - \pi_W y\|_* (\|u - \pi_V u\|_{\mathcal{L}} + |u_h - \pi_V u|_{S_p}) \end{aligned}$$

Remark 2. Note that for the GLS-method $\epsilon(h) = 0$ in (2.11) and (2.12). This follows from the fact that the whole residual is considered in the stabilisation term. This nice feature however only holds under exact quadrature. When the integrals are approximated, the quadrature error once again gives rise to oscillation terms from data that introduces a non-zero contribution to $\epsilon(h)$.

3.2. Continuous interior penalty. Since in this case we must account for possible oscillation of the physical coefficients we postpone the detailed analysis to the examples below and here only discuss the general principle. In this case we use $\gamma_{GLS} = 0$, $\gamma_{1,F} > 0$, $\gamma_{2,F} > 0$ in the general expressions for the stabilisation (3.4) and (3.5). The parameters $\gamma_{i,F}$, $i = 1, 2$ are stabilisation coefficients, the form of which will be problem specific and will be given for each problem below. The key observation is that the following discrete approximation result holds for suitably chosen $\gamma_{i,F}$ in $s_{cip}(\cdot, \cdot)$ (see [7, 8])

$$\|h^{\frac{1}{2}}(\beta_h \cdot \nabla u_h - I_{os}\beta_h \cdot \nabla u_h)\|^2 + \sum_K \|h(\Delta u_h - I_{os}\Delta u_h)\|^2 \le s_{cip}(u_h, u_h). \quad (3.10)$$

Here β_h is some piecewise affine interpolant of the velocity vector field β and I_{os} is the quasi-interpolation operator defined in each node of the mesh as a straight average of the function values from triangles sharing that node (see [8]). For example,

$$(I_{os}\Delta u_h)(x_i) = N_i^{-1} \sum_{\{K : x_i \in K\}} \Delta u_h(x_i)|_K,$$

with $N_i := \mathrm{card}\{K : x_i \in K\}$. Using (3.10) one may prove that

$$\inf_{v_h \in V_h} \|\mathcal{L}u_h - v_h\|_h \lesssim s_{cip}(u_h, u_h)^{\frac{1}{2}} + \epsilon(h)\|u_h\|. \quad (3.11)$$

It immediately follows that (3.2) and (3.3) are satisfied. We will leave the discussion of (2.11) - (2.14) and (3.11) to the applications below, giving the explicit form for $\epsilon(h)$ for each case. Here we instead proceed with an abstract analysis, ignoring the contribution from weakly imposed boundary conditions and assuming that all physical parameters are of order $O(1)$. Here we choose π_V and π_W as the L^2-projection in order to exploit orthogonality to "filter" the element residual. We let $\|\cdot\|_+$ and $\|\cdot\|_*$ have the same definition as in the GLS case and define

$$\|u\|_{\mathcal{L}} := \|h\mathcal{L}u\|_h + \|h^{\frac{1}{2}}\mu^{\frac{1}{2}}[\![\nabla u \cdot n_F]\!]\|_{\mathcal{F}_{int}} + \epsilon(h)\|u\|. \quad (3.12)$$

Then we proceed similarly as for GLS, but we use the orthogonality of the L^2-projection, ignoring here the contribution from boundary terms. It then follows using the orthogonality of the projection that formally

$$\begin{aligned} a_h(v - \pi_V v, x_h) &= \sum_{F \in \mathcal{F}_{int}} \langle u - \pi_V u, [\![\mu \nabla x_h \cdot n_F]\!]\rangle_F + \sum_K (u - \pi_V u, \mathcal{L}^* x_h - w_h)_K \\ &\le \|u - \pi_V u\|_+ (|x_h|_{S_a} + \epsilon(h)\|x_h\|). \end{aligned}$$

Similarly, to prove (2.12) we integrate by parts in the opposite direction in the second order operator and use the L^2-orthogonality to obtain

$$
\begin{aligned}
a_h(u-u_h, y-\pi_W y) &= \sum_{K\in\mathcal{T}_h} (\mathcal{L}(u-u_h), y-\pi_W y)_K \\
&\quad + \sum_{F\in\mathcal{F}_{int}} \langle [\![\mu\nabla u_h\cdot n_F]\!], y-\pi_W y\rangle_F \\
&= \sum_{K\in\mathcal{T}_h} (\mathcal{L}(u-\pi_V u)+\mathcal{L}(\pi_V u-u_h)-w_h, y-\pi_W y)_K \\
&\quad + \sum_{F\in\mathcal{F}_{int}} \langle [\![\mu\nabla u_h\cdot n_F]\!], y-\pi_W y\rangle_F \\
&\leq \|y-\pi_W y\|_*(\|h\mathcal{L}(u-\pi_V u)\|_h + |\pi_V u-u_h|_{S_p} + \epsilon(h)\|\pi_V u-u_h\|) \\
&\leq \|y-\pi_W y\|_*(\|u-\pi_V u\|_{\mathcal{L}} + |u_h-\pi_V u|_{S_p} + \epsilon(h)\|u-u_h\|)
\end{aligned}
$$

The last inequality follows by adding and subtracting u in the last norm in the right hand side to obtain $\epsilon(h)\|\pi_V u-u+u-u_h\|$. This term is then split using a triangular inequality and the approximation error integrated in the $\|\cdot\|_{\mathcal{L}}$ term. To use the L^2-projection in this fashion we must impose the boundary conditions weakly so that the boundary degrees of freedom are included in V_h. For the GLS-method one has the choice between weak and strong imposition of boundary condition. In the next section we will discuss how weakly imposed boundary conditions are included in the formulation.

3.3. Imposition of boundary conditions. The imposition of weak boundary conditions in this framework uses Nitsche type formulations. However they differ from the standard Nitsche boundary conditions in several ways:

- Both Dirichlet and Neumann conditions are imposed using penalty.
- There is no lower bound of the parameter for the imposition of Dirichlet type boundary conditions. This is related to the fact that the method never uses the coecivity of $a_h(\cdot,\cdot)$.
- Nitsche type boundary terms are added to $a_h(\cdot,\cdot)$ in order to ensure consistency and adjoint consistency, but the penalty is added to the operators $s_p(\cdot,\cdot)$ and $s_a(\cdot,\cdot)$, allowing for different boundary penalty for the primal and the adjoint. As we shall see below for some problems this is the only way to make the Nitsche formulation consistent.

If the primal and the dual problems have a Dirchlet boundary condition on Γ_D this is imposed by

$$
a_h(u_h,v_h) := a(u_h,v_h) - \langle \mu\nabla u_h, v_h\rangle_{\Gamma_D} - \langle \mu\nabla v_h, u_h\rangle_{\Gamma_D}
$$

and by adding the boundary penalty term

$$
\int_{\Gamma_D} \gamma_D \mu h^{-1} u_h v_h \,\mathrm{d}s \tag{3.13}
$$

to $s_p(\cdot,\cdot)$ and $s_a(\cdot,\cdot)$ with $\gamma_D>0$. In the non-homogeneous case the suitable data is added to the right hand side in the standard way. For Neumann conditions on Γ_N in both the primal and the adjoint problems, these are introduced in the standard way in $a(u,v)$ with a suitable modification of the right hand side of (2.1). No modification

is introduced in $a_h(\cdot,\cdot)$ but the following penalty is added to $s_p(\cdot,\cdot)$ and $s_a(\cdot,\cdot)$ with $\gamma_N > 0$

$$\int_{\Gamma_N} \gamma_N h \nabla u_h \cdot n \nabla v_h \cdot n \, \mathrm{d}s. \tag{3.14}$$

If the CIP method is used the only difference between $s_p(\cdot,\cdot)$ and $s_a(\cdot,\cdot)$ are possible difference in the boundary conditions. If the boundary conditions for u are non-homogeneous the usual data contributions are introduced in the right hand side $-s_p(u, w_h)$.

As mentioned in the introduction the semi-norm $|\cdot|_S$ can be a norm in certain situations so that the partial coercivity (2.10) implies the well-posedness of the linear system (2.5). In the following Proposition we discuss some basic sufficient conditions for the matrix to be invertible in the case of piecewise affine approximation spaces. For particular case other geometric arguments may prove fruitful as we shall see in the second example below.

Proposition 3.1. *The kernel of the linear system defined by* (2.5) *with the stabilisation* (3.6) *has dimension at most* $2(d+1)$ *for* $k = 1$. *The system* (2.5) *admits a unique solution if the boundary conditions satisfy one of the following conditions:*

1. *Two non-parallel polygon sides subject to Dirichlet boundary conditions.*
2. *Two non-orthogonal polygon sides subject one to a Dirichlet boundary condition and the other a Neumann condition imposed using* (3.14).
3. d *non-parallel polygon sides subject to Neumann conditions imposed using* (3.14) *and either* $1 \notin V_h$ *or there exists* $v_h, w_h \in V_h$ *such that* $a_h(1, v_h) \neq 0$ *and* $a_h(w_h, 1) \neq 0$.

Proof. It is immediate from (2.10) that the kernel of the system matrix of (2.5) can not be larger than the sum of the dimensions of the kernels of $s_p(\cdot,\cdot)$ and $s_a(\cdot,\cdot)$. For $s_{cip}(\cdot,\cdot)$ and $k = 1$ the kernel is identified as $[\mathbb{P}_1(\Omega)]^2$, with dimension $2(d+1)$.

To prove wellposedness of the linear system it is enough to prove uniqueness, we assume that $f = 0$ and prove that then $u_h \equiv z_h \equiv 0$.

If Dirichlet boundary condition is imposed on a boundary then the gradient must be zero in the tangential direction to this boundary, since the tangents of two boundaries spans $\mathbb{R}^d$ we conclude that $f = 0$ in (2.5) implies $u_h = 0$ due to (2.10) and similarly $z_h = 0$ and the matrix is invertible.

In the second case, the function is zero on the Dirichlet boundary and the gradient is zero in the tangential directions of the Dirichlet boundary condition eliminating d elements in the kernel. The penalty on the Neumann boundary, being non-orthogonal to the Dirichlet boundary, cancels the remaining free gradient. The same argument leads to both $u_h = 0$ and $z_h = 0$.

For the third case we observe that the term (3.14) acting on d non-parallel polygon sides implies that $\nabla u_h = 0$ and wellposedness is then immediate by the remaining conditions. □

Remark 3. Observe that the Proposition 3.1 holds for any bilinear form $a(\cdot,\cdot)$, *even strongly degenerate ones.*

4. Applications. We will now give three examples of problems that enter the abstract framework. The first two problems we introduce below have well-posed primal and adjoint problems so that the above theory applies. For each method we will propose a formulation and prove that the relations (2.7), (2.8) hold. We only consider the CIP-method in the examples below, in the first example we detail the dependence of physical parameters in all norms and coefficients. In the later examples

we assume that all physical parameters are unity and do not track the dependence. As suggested above we take $\pi_V \equiv \pi_W \equiv \pi_L$. In each case we will detail the form of $\epsilon(h)$. In the last case, the elliptic Cauchy problem, the stability properties of the problem is strongly depending on the geometry of the problem and the assumption of well-posedness does not hold in general. We will nevertheless propose a method that satisfies the assumptions of the general theory and then study its performance numerically.

4.1. Nonsymmetric indefinite elliptic problems. Our first examples consist of a convection-diffusion–reaction problem with non-solenoidal velocity field as is the case for reactive transport in compressible flow. We first consider the case of homogeneous Dirichlet conditions where the analysis of [23] applies. Then we consider the case where failure of the coercivity is due also to the boundary condition, here we study a convection-diffusion equation with homogeneous Neumann boundary conditions. We will detail only how the analysis of this case differs from the Dirichlet case. For a detailed analysis of the well-posedness of the continuous problems we refer to [14, 16] and for a finite element analysis in the case of homogeneous Neumann conditions to [10]. Recent work on numerical methods for these problems have focused on finite volume methods, see [15, 11] or hybrid finite element finite volume methods [19].

4.1.1. Reactive transport in compressible flow: Dirichlet conditions. In combustion problems for example it is important to accurately compute the transport of the reacting species in the compressible flow. We suggest a scalar model problem of convection-diffusion type with a linear reaction term cu, where the reaction can have arbitrary sign.

$$\begin{aligned} \mathcal{L}u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

The dual adjoint takes the form

$$\begin{aligned} \mathcal{L}^* z := -\mu\Delta z - \beta\cdot\nabla z + cz &= g \quad \text{in } \Omega \\ z &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.2}$$

The variational formulation is obtained by taking $V := H^1_0(\Omega)$ and introducing the bilinear form

$$a(u,v) := (\mu\nabla u, \nabla v) + (\nabla\cdot(\beta u) + cu, v).$$

We assume that $f, g \in L^2(\Omega)$, that both (4.1) and (4.2) are well posed in $H^1_0(\Omega)$ by the Fredholm alternative and that the smoothing property (2.3) holds. See [14] for an analysis of existence and uniqueness under weaker regularity assumptions on β and c, with $c \geq 0$. The below analysis can also be carried out assuming less regularity, but the constraints on the mesh-size for the error estimate to hold will be stronger. Recall that the constants in the estimate (2.3) also depends on the regularity of the coefficients.

The discrete form of the bilinear form is given by

$$a_h(u_h, v_h) := a(u_h, v_h) - \langle \nabla u_h\cdot n, v_h\rangle_{\partial\Omega} - \langle \nabla v_h \cdot n, u_h\rangle_{\partial\Omega} - \langle (\beta\cdot n)_- u_h, v_h\rangle_{\partial\Omega} \tag{4.3}$$

where $(\beta\cdot n)_\pm := \frac{1}{2}(\beta\cdot n \pm |\beta\cdot n|)$. The stabilisation is chosen as

$$s_p(u_h, v_h) := s_{cip}(u_h, v_h) + s^-_{bc}(u_h, v_h) \tag{4.4}$$

and

$$s_a(z_h, v_h) := s_{cip}(z_h, v_h) + s_{bc}^+(z_h, v_h) \tag{4.5}$$

where $\gamma_{1,F} \sim (\mu + \|\beta_h \cdot n_F\|_{\infty,F} h_F)$ and $\gamma_{2,F} \sim \mu$ in (3.6) with β_h the nodal interpolant of β and

$$s_{bc}^{\pm}(x_h, v_h) := \left\langle \mu h^{-1}\, x_h, v_h \right\rangle_{\partial\Omega} + \langle |(\beta \cdot n)_\pm| x_h, v_h \rangle_{\partial\Omega}. \tag{4.6}$$

Proposition 4.1. *(Existence of discrete solutions) Let $k = 1$. Then the formulation* (2.5) *with the bilinear form* (4.3) *and the stabilisation* (4.4)–(4.5) *admits a unique solution $u_h \in V_h$.*

Proof. Immediate by Proposition 3.1. ∎

It is well known that the bilinear form (4.3) satisfies the consistency relations (2.7) and (2.8) and that the stabilisation (4.4)-(4.5) satisfies the consistency properties (2.13), (2.14) and (2.15). Now we define the norms by

$$\|v\|_+ := \|v\|_* := \|\mu^{\frac{1}{2}} h^{-\frac{1}{2}} v\|^2_{\mathcal{F}_{int}} + \|(\zeta_{Pe} + \zeta_{Osc} + \zeta_0) v\| + \|h^{\frac{1}{2}} \mu^{\frac{1}{2}} \nabla v\|_{\partial\Omega} + \|\sigma^{\frac{1}{2}} v\|_{\partial\Omega}.$$

Here $\zeta_{Pe} := (\sigma^{\frac{1}{2}} h^{-\frac{1}{2}} + \mu^{\frac{1}{2}} h^{-1})$, with $\sigma := \|\beta\|_{L^\infty(\Omega)}$, $\zeta_{Osc} := h^{-\frac{1}{2}} (|\beta|_{W^{2,\infty}} + |c|_{W^{1,\infty}})^{\frac{1}{2}}$ and $\zeta_0 = (\|\nabla \beta\|_{L^\infty} + \|c\|_{L^\infty})^{\frac{1}{2}}$. Also define

$$\begin{aligned}\|v\|_{\mathcal{L}} &:= \|\mu^{\frac{1}{2}} h \Delta v\|_h + \|\sigma^{-\frac{1}{2}} h^{\frac{1}{2}} \beta \cdot \nabla v\| + \||c|^{\frac{1}{2}} v\| \\ &\quad + \|\mu^{\frac{1}{2}} h^{\frac{1}{2}} [\![\nabla v \cdot n_F]\!]\|_{\mathcal{F}_{int}} + \|(\mu^{\frac{1}{2}} h^{-\frac{1}{2}} + \sigma^{\frac{1}{2}}) v\|_{\partial\Omega} + |v|_{S_p} + \epsilon(h)\|v\|.\end{aligned}$$

It is straightforward to show that

$$\|u - \pi_V u\|_+ + \|u - \pi_V u\|_{\mathcal{L}} \lesssim (\zeta_{Pe} + \zeta_{Osc} + \zeta_0) h^{k+1} |u|_{H^{k+1}(\Omega)}$$

and similarly for $\|u - \pi_W u\|_*$.

It then only remains to prove the continuities (2.11) and (2.12) to conclude that the Theorem 2.2 holds.

Proposition 4.2. *The bilinear form* (4.3) *satisfies the continuities* (2.11) *and* (2.12) *with*

$$\epsilon(h) \sim h^{\frac{3}{2}} (|\beta|_{W^{2,\infty}} + |c|_{W^{1,\infty}})^{\frac{1}{2}}$$

.

Proof. First we consider (2.11). After an integration by parts in $a(\cdot,\cdot)$ we have

$$\begin{aligned}a_h(u - \pi_V u, x_h) = \sum_{F \in \mathcal{F}_{int}} \langle u - \pi_V u, [\![\mu \nabla x_h \cdot n_F]\!] \rangle_F + \sum_K (u - \pi_V u, \mathcal{L}^* x_h)_K \\ - \langle u - \pi_V u, (\beta \cdot n)_+ x_h \rangle_{\partial\Omega} = I + II + III.\end{aligned}$$

Considering $I - III$ we find using Cauchy-Schwarz inequality

$$I + III \leq \|u - \pi_V u\|_+ |x_h|_{S_a}.$$

For II, using the discrete interpolation results (3.10), the discrete commutator property, see [3], and standard approximation followed by an inverse inequality in the last

term

$$
\begin{aligned}
II &= \sum_K (u - \pi_V u, -i_h\beta \cdot \nabla x_h + I_{os}(i_h\beta_h \cdot \nabla x_h) - \mu\Delta x_h + I_{os}\mu\Delta x_h)_K \\
&\quad + (u - \pi_V u, c x_h - i_h(c x_h)) + (u - \pi_V u, (\beta - i_h\beta)\cdot\nabla x_h) \\
&\leq c_{os}\|u - \pi_V u\|_+ (|x_h|_{S_a} + c_i h^{\frac{3}{2}}(|\beta|_{W^{2,\infty}} + |c|_{W^{1,\infty}})^{\frac{1}{2}}\|x_h\|).
\end{aligned}
$$

The second continuity follows in a similar fashion,

$$
\begin{aligned}
a_h(u - u_h, y - \pi_W y) &= (\mathcal{L}(u - u_h), y - \pi_W y)_h + \sum_{F\in\mathcal{F}_{int}} \langle [\![\mu\nabla u_h \cdot n_F]\!], y - \pi_W y\rangle_F \\
&\quad + \langle(\beta\cdot n)_- u_h, y - \pi_W y\rangle_{\partial\Omega} + \langle \mu\nabla(y - \pi_W y)\cdot n, u_h\rangle_{\partial\Omega} \\
&= I + II + III + IV.
\end{aligned}
$$

Considering first the term I we get, with $\xi_h = \pi_V u - u_h$

$$
\begin{aligned}
I &= (\mathcal{L}(u - \pi_V u) + \mathcal{L}\xi_h, y - \pi_W y) \\
&\lesssim \|y - \pi_W y\|_* (\|u - \pi_V u\|_{\mathcal{L}} + |\xi_h|_{S_p} + h^{\frac{3}{2}}(|\beta|_{W^{2,\infty}(\Omega)} + |c|_{W^{1,\infty}(\Omega)})^{\frac{1}{2}}\|u - u_h\|),
\end{aligned}
$$

where we used once again the inequalities

$$
(\mu\Delta\xi_h + \beta_h\nabla\xi_h, y - \pi_W y) \leq c_{os}|\xi_h|_{S_p}\|y - \pi_W y\|_*,
$$

$$
((\beta - i_h\beta)\cdot\nabla\xi_h, y - \pi_W y) \lesssim h|\beta|_{W^{2,\infty}}(\|u - \pi_V u\| + \|u - u_h\|)\|y - \pi_W y\|
$$

and

$$
\begin{aligned}
((\nabla\cdot\beta + c)\xi_h, y - \pi_W y) &= ((\nabla\cdot\beta + c)\xi_h - i_h((\nabla\cdot\beta + c)\xi_h), y - \pi_W y) \\
&\lesssim h(|\beta|_{W^{2,\infty}} + |c|_{W^{1,\infty}})(\|u - \pi_V u\| + \|u - u_h\|)\|y - \pi_W y\|.
\end{aligned}
$$

In the last inequality we once again applied then discrete commutator property. For the second term and third terms we have using the Cauchy-Schwarz inequality, recalling the form of the boundary penalty term,

$$
II + III + IV \lesssim |u_h|_{S_p}\|y - \pi_W y\|_*.
$$

Hence, by adding and subtracting $\pi_V u$ in the $|u_h|_{S_p}$ contribution and using approximation we conclude that the claim holds with $\epsilon(h) \sim h^{\frac{3}{2}}(|\beta|_{W^{2,\infty}} + |c|_{W^{1,\infty}})^{\frac{1}{2}}$. □

Remark 4. Note that if the physical parameters are constant, then the analysis holds without restrictions on the mesh size in contrast to the standard Galerkin analysis of [23].

Remark 5. There is some flexibility in the choice of norms in the analysis above. Here we have tried to mimic the analysis of a stabilised method that is robust for high Péclet number, however this estimate is not robust, even if we assume that u is as smooth as we like. This is due to the use of a Nitsche type trick where the actual regularity constant of the estimate (2.3) *appears. If the dependence on physical parameters in Theorem 2.2 is taken into account the upper bound typically reads for* $k = 1$,

$$
\|u - u_h\| \lesssim c_{a,\Omega}(\zeta_{Pe} + \zeta_{Osc} + \zeta_0)^2 h^4 |u|_{H^2(\Omega)}.
$$

Assuming that all physical parameters except μ are $O(1)$ we get

$$\|u - u_h\| \lesssim c_{a,\Omega} \left(\frac{1}{h} + \frac{\mu}{h^2} \right) h^4 |u|_{H^2(\Omega)}$$

Here the constant $c_{a,\Omega}$ typically is proportional to some negative power of μ, making the estimate valid only for moderate Péclet numbers. If we assume that $c_{a,\Omega} = O(\mu^{-1})$ we see that the quasi optimal convergence of order $h^{\frac{3}{2}}$ is obtained when $h^{\frac{3}{2}} < \mu$. A more precise estimate for the hyperbolic regime is the subject of the second part of this work [6].

4.1.2. Transport in compressible flow: pure Neumann conditions. We will now consider the convection–diffusion equation with homogeneous Neumann conditions. The main difficulty in this problem compared to the previous one is that due to the homogeneous Neumann condition, the primal and dual problems have different boundary conditions. The non-solenoidal β imposes special compatibility conditions on g leading to complications in the finite element analysis and additional stability issues for the discrete solution. For this example we will assume that all physical parameters are order one. After having presented the problem and the method we propose, we first show that the the discrete problem is well-posed for all mesh-sizes when piecewise affine approximation is used. Then we prove that the assumpitions of Lemma 2.1 and Theorem 2.2 are satisfied. Optimal error estimates for the problem similar to that above are obtained after accounting for some minor modifications needed to accomodate the compatibility conditions particular to this problem. The problem reads

$$\begin{aligned} -\Delta u + \nabla \cdot (\beta u) &= f \quad \text{in } \Omega \\ -\nabla u \cdot n + \beta \cdot n u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.7}$$

The dual adjoint problem is formally written

$$\begin{aligned} -\Delta z - \beta \cdot \nabla z &= g \quad \text{in } \Omega \\ -\nabla z \cdot n &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.8}$$

We assume that the following compatibility conditions hold

$$\int_\Omega f \, \mathrm{d}x = 0, \quad \int_\Omega gm \, \mathrm{d}x = 0 \tag{4.9}$$

where $m \in H^2(\Omega)$, $m > 0$ is the unique solution to the homogeneous form of the primal problem

$$\begin{aligned} -\Delta m + \nabla \cdot (\beta m) &= 0 \quad \text{in } \Omega \\ -\nabla m \cdot n + \beta \cdot n m &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.10}$$

under the additional constraint

$$|\Omega|^{-1} \int_\Omega m \, \mathrm{d}x = 1.$$

Then the problems (4.7) and (4.8) are both well-posed by the Fredholm alternative. Since we assume that the regularity estimate (2.3) holds, $m \in C^0(\bar{\Omega})$ and $\sup_{x\in\Omega} m =: M \in \mathbb{R}^+$ and since $m > 0$ we may introduce $m_{min} := inf_\Omega \, m > 0$ (see [10]).

The problem is cast in the form (2.1) by setting $V := H^1(\Omega) \cap L^2_0(\Omega)$, where $L^2_0(\Omega)$ denotes the set of functions with global average zero, and by taking

$$a(u, v) := (\nabla u, \nabla v) - (u, \beta \cdot \nabla v).$$

The finite element method (2.5) is obtained by setting $V_h := W_h := X_h^k \cap L^2_0(\Omega)$

$$a_h(u_h, v_h) := a(u_h, v_h) \tag{4.11}$$

and the stabilisation operators

$$s_x(\cdot,\cdot) := s_{ip}(\cdot,\cdot) + s_{bc,x}(\cdot,\cdot), \text{ with } x = a, p. \tag{4.12}$$

$s_{cip}(\cdot,\cdot)$ is given by (3.6) with $\gamma_{i,F} := 1$, $i = 1, 2$. The boundary operators finally are defined by

$$s_{bc,p}(u_h, v_h) := \int_\Omega h(\nabla u_h \cdot n - \beta \cdot n u_h)(\nabla v_h \cdot n - \beta \cdot n v_h) \; \mathrm{d}s$$

for $k \geq 2$ and

$$s_{bc,p}(u_h, v_h) := \int_\Omega h(\nabla u_h \cdot n - (i_h\beta) \cdot n u_h)(\nabla v_h \cdot n - (i_h\beta) \cdot n v_h) \; \mathrm{d}s$$

for $k = 1$, $s_{bc,a}(\cdot,\cdot)$ finally is given by equation (3.14), with $\gamma_N = 1$. The boundary stabilisation operator for $k = 1$ is only weakly consistent. It is straightforward to show that the inconsistency introduced by replacing β by $i_h\beta$ is compatible with (2.13). We omit the details here, but similar arguments are used below to prove the continuity (2.12).

Proposition 4.3. *(Existence of discrete solution) Assume $k = 1$ in the definition of V_h. Then there exists a unique solution u_h to the discrete problem* (2.5).

Proof. As before we assume $f = 0$ and observe that

$$s_p(u_h, u_h) + s_a(z_h, z_h) = 0.$$

This implies $z_h, u_h \in \mathbb{P}_1(\Omega)$. Since $\|\nabla z_h \cdot n\|_{\partial\Omega} = 0$ and $s_{av}(z_h, z_h) = 0$, we conclude that $z_h = 0$. For u_h there holds

$$\|\nabla u_h \cdot n + (i_h\beta \cdot n)u_h\|_{\partial\Omega} = 0.$$

Since $\nabla u_h \cdot n$ is constant on every polyhedral side Γ of Ω so is $(i_h\beta \cdot n)u_h$. But since $(i_h\beta \cdot n)u_h|_\Gamma \in \mathbb{P}_2(\Gamma)$ we conclude that both $i_h\beta$ and u_h must be constant. Since this is true for all faces Γ of Ω, u_h is a constant globally. We conclude by recalling that zero average was inposed on the approximation space. ∎

In case $k \geq 2$, we let the norms $\|\cdot\|_+$, $\|\cdot\|_*$ be defined by (3.9) and $\|\cdot\|_{\mathcal{L}}$ by (3.12). When $k = 1$ we let the norm $\|\cdot\|_+$ be defined by (3.9), but define

$$\|v\|_* := \|h^{-1}v\| + \|h^{-\frac{1}{2}}v\|_{\mathcal{F}} + \epsilon(h)\|v\|$$

and

$$\|v\|_{\mathcal{L}} := \|\mathcal{L}v\|_h + \|h^{\frac{1}{2}}[\![\nabla v \cdot n_F]\!]\|_{\mathcal{F}_{int}} + \|h^{\frac{1}{2}}\nabla v \cdot n\|_{\partial\Omega} + (\epsilon(h) + \sigma)\|h^{\frac{1}{2}}v\|_{\partial\Omega} + \epsilon(h)\|v\|.$$

For the projection operators π_V and π_W we once again choose the L^2-projection.

Proposition 4.4. *Assume*

$$\epsilon(h) \sim h^2 |\beta|_{W^{2,\infty}}.$$

Then the bilinear form (4.11) *satisfies the continuities* (2.11) *and* (2.12).

Proof. As before we integrate by parts in $a_h(\cdot,\cdot)$ to obtain

$$a_h(u - \pi_V u, x_h) = \sum_{F \in \mathcal{F}_{int}} \langle u - \pi_V u, [\![\nabla x_h \cdot n_F]\!] \rangle_F + \sum_K (u - \pi_V u, \mathcal{L}^* x_h)_K$$

$$+ \langle u - \pi_V u, \nabla x_h \cdot n \rangle_{\partial\Omega} = I + II + III.$$

The treatment of terms I and II are identical to the Dirichlet case. Term III is bounded using Cauchy-Schwarz inequality, recalling that the Neumann condition is penalised in $s_a(\cdot,\cdot)$

$$III \leq \|u - \pi_V u\|_+ |x_h|_{S_a}.$$

The second continuity follows in a similar fashion. We write

$$\begin{aligned} a_h(u - u_h, y - \pi_W y) = &\sum_{K \in \mathcal{T}_h} (\mathcal{L}(u - u_h), y - \pi_W y)_K \\ &+ \sum_{F \in \mathcal{F}_{int}} \langle [\![\nabla u_h \cdot n_F]\!], y - \pi_W y \rangle_F \\ &- \langle y - \pi_W y, \nabla u_h \cdot n - (\beta \cdot n) u_h \rangle_{\partial\Omega} \\ = &\, I + II + III \end{aligned}$$

and observe that the treatment of terms I and II is analoguous with the Dirichlet case. Recalling that $\nabla u_h \cdot n - (\beta \cdot n) u_h$ is penalised in $s_p(\cdot,\cdot)$, we may conclude as before using a Cauchy-Schwarz inequality when $k \geq 2$.

$$III \leq \|u - \pi_V u\|_* |u_h|_{S_p}.$$

For the case $k = 1$ we must take care to handle the lack of consistency. Therefore we add and subtract $i_h \beta$ and use the boundary condition on u to get

$$\begin{aligned} III = &\langle y - \pi_W y, \nabla(u_h - u) \cdot n - (i_h \beta \cdot n)(u_h - u) \rangle_{\partial\Omega} \\ &+ \langle y - \pi_W y, (i_h \beta - \beta) \cdot n (u_h - u) \rangle_{\partial\Omega}. \end{aligned}$$

First we add and subtract $\pi_V u$, so that $u - u_h = u - \pi_V u + \xi_h$, $\xi_h := \pi_V u - u_h$ and split the scalar products with Cauchy-Schwarz inequality. Then we apply a trace inequality in the boundary norm on ξ_h, and use the definition of the norms, in particular that $\|h^{-\frac{1}{2}}(y - \pi_W y)\|_{\partial\Omega} \leq \|y - \pi_W y\|_*$ and apply approximation in L^∞ for β to obtain

$$III \lesssim \|y - \pi_W y\|_* (\|u - \pi_V u\|_{\mathcal{L}} + |\xi_h|_{S_p}) + \|y - \pi_W y\|_{\partial\Omega} h^{\frac{3}{2}} |\beta|_{W^{2,\infty}} \|\xi_h\|.$$

Using a triangular inequality in $\|\xi_h\|$, after having added and subtracted u, we obtain

$$III \lesssim \|y - \pi_W y\|_* (\|u - \pi_V u\|_{\mathcal{L}} + |\xi_h|_{S_p}) + \|y - \pi_W y\|_{\partial\Omega} h^{\frac{3}{2}} |\beta|_{W^{2,\infty}} \|u - u_h\|.$$

Using the definition of the norms we see that the last term in the right hand side may be written

$$\begin{aligned}\|y - \pi_W y\|_{\partial\Omega} h^{\frac{3}{2}} |\beta|_{W^{2,\infty}} \|u - u_h\| &\leq \|y - \pi_W y\|_* h^2 |\beta|_{W^{2,\infty}} \|u - u_h\| \\ &\sim \|y - \pi_W y\|_* \epsilon(h) \|u - u_h\|.\end{aligned}$$

The proof is complete. □

Remark 6. It is straightforward to verify that Lemma 2.1 holds. The assumptions of Theorem 2.2, however still are not satisfied since we want to use the solutions of the problems $\mathcal{L}^*\varphi = u - u_h$ *and* $\mathcal{L}\psi = z_h$*, but the solution* φ *will in general not exist since* $u - u_h$ *does not satisfy the second compatibility condition of equation* (4.9)*. Instead we will use* m*, the solution of* (4.10) *as weight, as suggested in [10], and solve the wellposed problem*

$$\mathcal{L}^*\varphi = (u - u_h)/m.$$

We may then write

$$\|(u - u_h)m^{-\frac{1}{2}}\|^2 + \|z_h\|^2 = (u - u_h, \mathcal{L}^*\varphi) + (\mathcal{L}\psi, z_h)$$

and proceed as in Theorem 2.2, now using the stability estimate

$$|\varphi|_{H^2(\Omega)} \leq c_{a,\Omega} \|(u - u_h)m^{-1}\| \leq c_{a,\Omega}/m_{min}^{\frac{1}{2}} \|(u - u_h)m^{-1/2}\|$$

to obtain

$$\|(u - u_h)m^{-\frac{1}{2}}\| + \|z_h\| \lesssim h^{k+1} |u|_{H^{k+1}(\Omega)}.$$

For an estimate in the unweighted L^2*-norm we observe that*

$$M^{-\frac{1}{2}} \|u - u_h\| \leq \|(u - u_h)m^{-\frac{1}{2}}\|.$$

Convergence follows by Lemma 2.1 and the modified Theorem 2.2. Observe that the constants in $\epsilon(h)$ *now depends on the (unknown) minimum value of* m*.*

Remark 7. In practice the zero average condition can be imposed using Lagrange multipliers. The above analysis holds for that case after minor modifications.

4.2. The Cauchy problem. We consider the case of a Helmholtz-type problem where both the solution itself and its normal gradient is specified on one portion of the domain and the other portion is free. We let Γ_V and Γ_W be connected subsets of $\partial\Omega$ such that $\partial\Omega := \Gamma_V \cup \Gamma_W$ and $\Gamma_V \cap \Gamma_W = \emptyset$. We will consider the problem, $\kappa \in \mathbb{R}$,

$$\begin{aligned}-\Delta u + \kappa u &= f \text{ in } \Omega \\ \nabla u \cdot n = u &= 0 \text{ on } \Gamma_V,\end{aligned} \tag{4.13}$$

with dual problem

$$\begin{aligned}-\Delta z + \kappa z &= g \text{ in } \Omega \\ \nabla z \cdot n = z &= 0 \text{ on } \Gamma_W.\end{aligned} \tag{4.14}$$

The weak formulations (2.1) and (2.2) are obtained by setting

$$V := \{v \in H^1(\Omega) : v|_{\Gamma_V} = 0\}$$

and

$$W := \{v \in H^1(\Omega) : v|_{\Gamma_W} = 0\}$$

and defining

$$a(u, v) := (\nabla u, \nabla v) + \kappa(u, v), \quad \forall u \in V,\, v \in W.$$

Note that both symmetry and Gårdings inequality fail in this case because the functions in the bilinear form have to be taken in different spaces and hence the choice $v = u$ is prohibited.

To design a suitable discrete formulation (2.5) for this problem we generalise the ideas of the Nitsche type weak imposition of boundary condition. Observe that in this case boundary conditions imposed using penalty in the standard fashion can not be consistent for both the primal and the adjoint problems, since the primal and dual solution are zero on different parts of the boundary. It is therefore important in this case that two stabilisation operators are used, one for the primal and one for the adjoint. We propose the bilinear form $a_h(\cdot, \cdot) : V_h \times V_h \to \mathbb{R}$ defined as

$$a_h(u_h, v_h) := a(u_h, v_h) - \langle \nabla v_h \cdot n, u_h \rangle_{\Gamma_V} - \langle \nabla u_h \cdot n, v_h \rangle_{\Gamma_W} \tag{4.15}$$

and for the stabilisation we use

$$s_x(u_h, v_h) := s_{cip}(u_h, v_h) + s_{bc,x}(u_h, v_h), \quad x = a, p \tag{4.16}$$

where $s_{cip}(\cdot, \cdot)$ is given by (3.6), with $\gamma_{F,i} := 1$, $i = 1, 2$, and

$$s_{bc,x}(u_h, v_h) := \int_X (h^{-1} u_h\, v_h + h \nabla u_h \cdot n \nabla v_h \cdot n)\, \mathrm{d}s,$$

where $X = \Gamma_V$ for $x = p$ and $X = \Gamma_W$ for $x = a$. If some part of the boundary is equipped with Dirichlet or Neumann boundary conditions this is imposed as described in Section 3.3.

Proposition 4.5. *(Existence of discrete solution for $k = 1$) Define* (2.5) *by the bilinear forms* (4.15) *and* (4.16). *Let $k = 1$ in V_h. Then there exists a unique solution $(u_h, z_h) \in [V_h]^2$ to* (2.5).

Proof. Let $f = 0$, by (2.10) there holds, $u_h, z_h \in \mathbb{P}_1(\Omega)$ and $u_h|_{\Gamma_V} = \nabla u_h \cdot n|_{\Gamma_V} = 0$ as well as $z_h|_{\Gamma_W} = \nabla z_h \cdot n|_{\Gamma_W} = 0$, by which we conclude that the matrix is invertible using case (2) of Proposition 3.1. ◻

For the error analysis we once again choose the interpolants π_V and π_W to be the standard L^2-projection π_L. We will now prove that the assumptions (2.7)-(2.8) and (2.11)-(2.12) are satisfied.

Lemma 4.6. *(Consistency of bilinear form) The bilinear form* (4.15) *satisfies* (2.7) *and* (2.8).

Proof. By an integration by parts we see that for u solution of (4.13)

$$\begin{aligned}(-\Delta u + \kappa u, v + v_h) = (\nabla u, \nabla(v + v_h)) - \underbrace{\langle \nabla u \cdot n, v + v_h \rangle_{\Gamma_W}}_{\text{since } \nabla u \cdot n = 0 \text{ on } \Gamma_V} - \underbrace{\langle \nabla(v + v_h) \cdot n, u \rangle_{\Gamma_V}}_{\text{since } u = 0 \text{ on } \Gamma_V} \\ + (\kappa u, v + v_h) = a_h(u, v + v_h).\end{aligned}$$

Similarly for z solution of (4.14) consistency follows by observing that

$$(v + v_h, -\Delta z) = (\nabla(v + v_h), \nabla z) - \underbrace{\langle \nabla z \cdot n, v + v_h \rangle_{\Gamma_V}}_{\text{since } \nabla z \cdot n = 0 \text{ on } \Gamma_W} - \underbrace{\langle \nabla(v + v_h) \cdot n, z \rangle_{\Gamma_W}}_{\text{since } z = 0 \text{ on } \Gamma_W}.$$

$\square$

We define the norms $\|\cdot\|_+$ and $\|\cdot\|_*$ by

$$\|v\|_+ := \|h^{-\frac{1}{2}}v\|_{\mathcal{F}_{int}} + \|h^{-1}v\| + \|h^{-\frac{1}{2}}v\|_{\Gamma_W} + \|h^{\frac{1}{2}}\nabla v\cdot n\|_{\Gamma_W},$$

$$\|v\|_* := \|h^{-\frac{1}{2}}v\|_{\mathcal{F}_{int}} + \|h^{-1}v\| + \|h^{-\frac{1}{2}}v\|_{\Gamma_V} + \|h^{\frac{1}{2}}\nabla v\cdot n\|_{\Gamma_V}.$$

and

$$\|v\|_{\mathcal{L}} := \|h\mathcal{L}v\|_h + \|h^{\frac{1}{2}}[\![\nabla v\cdot n_F]\!]\|_{\mathcal{F}_{int}} + \|h^{-\frac{1}{2}}v\|_{\Gamma_V} + \|h^{\frac{1}{2}}\nabla v\cdot n\|_{\Gamma_V}.$$

It is straightforward to show (2.13) and (2.14).

PROPOSITION 4.7. *For $a_h(\cdot,\cdot)$ defined by* (4.15)*, the continuities* (2.11) *and* (2.12) *hold with $\epsilon(h) = 0$.*

Proof. We proceed as before using an integration by parts in (4.15) to obtain

$$\begin{aligned} a_h(v-\pi_V v, x_h) &= \sum_{F\in\mathcal{F}_{int}} \langle v-\pi_V v, [\![\nabla x_h\cdot n_F]\!]\rangle_F + \sum_K (v-\pi_V v, -\Delta x_h + \kappa x_h)_K \\ &+ \langle v-\pi_V v, \nabla x_h\cdot n\rangle_{\Gamma_W} - \langle \nabla(v-\pi_V v)\cdot n, x_h\rangle_{\Gamma_W} = I + II + III + IV. \end{aligned}$$

The first sum I is upper bounded as before using the Cauchy-Schwarz inequality and for the second sum, we use the orthogonality of the L^2-projection, $(v-\pi_V v, \kappa x_h) = 0$ and the discrete interpolation inequality (3.10) leading to

$$I + II \lesssim \|u-\pi_V u\|_+ |x_h|_{S_a}.$$

For the terms III and IV we note that by the definition of $\|\cdot\|_+$ and $|\cdot|_{S_a}$ there also holds

$$III + IV \leq \|u-\pi_V u\|_+ |x_h|_{S_a}.$$

This ends the proof of (2.11). The proof of (2.12) is similar. Using integration by parts in the other direction we have

$$\begin{aligned} a_h(u-u_h, y-\pi_W y) = &\sum_K (-\Delta(u-u_h) + \kappa(u-u_h), y-\pi_W y)_K \\ &+ \sum_{F\in\mathcal{F}_{int}} \langle [\![\nabla u_h\cdot n_F]\!], y-\pi_W y\rangle_F + \langle u-u_h, \nabla(y-\pi_W y)\cdot n\rangle_{\Gamma_V} \\ &+ \langle \nabla(u-u_h)\cdot n, y-\pi_W y\rangle_{\Gamma_V} = I + II + III + IV. \end{aligned}$$

Using the same arguments as before, adding and subtracting $\pi_V u$ in all the terms in the left slot we have for the term I, using $\xi_h = \pi_V u - u_h$,

$$\begin{aligned} I &= (-\Delta(u-\pi_V u) + \kappa(u-\pi_V u), y-\pi_W y)_h - (\Delta\xi_h - I_{os}\Delta\xi_h, y-\pi_W y)_h \\ &\lesssim (\|u-\pi_V u\|_{\mathcal{L}} + |\xi_h|_{S_p})\|y-\pi_W y\|_*. \end{aligned}$$

By the definition of the stabilisation operator and the fact that $u = \nabla u\cdot n = 0$ on Γ_V we may once again add and subtract $\pi_V u$ in the terms II and III to obtain

$$\begin{aligned} II + III + IV &= \langle [\![\nabla u_h\cdot n_F]\!], y-\pi_W y\rangle_{\mathcal{F}_{int}} \\ &\quad - \langle u_h, \nabla(y-\pi_W y)\cdot n\rangle_{\Gamma_V} - \langle \nabla u_h\cdot n, y-\pi_W y\rangle_{\Gamma_V} \\ &\leq \|y-\pi_W y\|_*(\|u-\pi_V u\|_{\mathcal{L}} + |\xi_h|_{S_p}). \end{aligned}$$

By which we conclude. ∎

Corollary 4.8. *Assume that the problems* (4.13) *and* (4.14) *admit unique solutions for which* (2.3) *holds. Then the conclusions of Theroem 2.2 holds for* (2.5) *defined by* V_h, (4.15) *and* (4.16).

Proof. In Lemma 4.6 we verified the consistencies (2.7) and (2.8). In Proposition 4.7 we verified the continuities (2.11) and (2.12). It is straightforward to verify that (2.13)-(2.14) hold for $\pi_V = \pi_W = \pi_L$ and $s_p(\cdot,\cdot)$, $s_a(\cdot,\cdot)$ defined by (4.16) under the assumptions on the mesh and the regularity assumptions on the solution. ∎

Remark 8. Admittedly Corollary 4.8 is of purely academic interest since the Cauchy problem under consideration in general is ill-posed, with very weak stability properties. As we shall see in the numerical section the method nevertheless returns useful approximations. An example of a sufficient condition for Theorem 2.2 to result in a convergence estimate is that there exists $M \in \mathbb{R}^+$ *and* $s \in \mathbb{R}$, *with* $s > -k/2$ *such that* $\|\nabla z\| \le M h^s$, *for all* $u - u_h$. *The expected convergence rate in that case would be*

$$\|u - u_h\| \lesssim h^{k/2+s} |u|_{H^{k+1}(\Omega)}, \tag{4.17}$$

however since $\|\nabla z\|$ *can be expected to vary significantly for different realisations* u_h *the convergence must be expected to be very irregular, even if* (4.17) *holds. Unfortunately, no such stability estimates are known for the Cauchy problem, we refer to [4] for conditional stability estimates for the problem* (4.13) *in general Lipschitz domains and to [17, 22] and [21] for other work on finite element methods on the Cauchy problem and some stability results under special geometrical assumptions.*

5. Numerical investigations. We will present numerical examples of convergence for a smooth exact solution of the applications given above. For the computations we have used FreeFEM++, [18]. All problems will be set in $\Omega := (0,1) \times (0,1)$. We consider an example given in [11] where, in (4.1), the physical parameters are chosen as$\mu = 1$, $c = 0$,

$$\beta := -100 \begin{pmatrix} x + y \\ y - x \end{pmatrix}$$

and the exact solution is given by

$$u(x,y) = 30x(1-x)y(1-y). \tag{5.1}$$

This function satisfies homogeneous Dirichlet boundary conditions and has $\|u\| = 1$. Note that $\|\beta\|_{L^\infty} = 200$ and $\nabla \cdot \beta = -200$, making the problem strongly noncoercive with a medium high Péclet number. The right hand side is then chosen as $\mathcal{L}u$ and in the case of (non-homogeneous) Neumann conditions, a suitable right hand side is introduced to make the boundary penalty term consistent. We use unstructured meshes with 2^N elements on each side, $N = 3, \dots, 8$ and, drawing on our previous experience of the CIP-method we fix the stabilisation parameters to be $\gamma_{1,F} = 0.01$ for piecewise affine approximation and $\gamma_{i,F} = 0.001$, $i = 1, 2$ for piecewise quadratic approximation. The boundary penalty parameter is chosen to be $\gamma_{bc} = 10$ for both cases and both for Dirichlet and Neumann penalty terms. Let us remark that in particular for the ill-posed Cauchy problem, an optimal choice of the stabilisation parameter can make a big impact on the error on a fixed mesh, but does not appear to influence the convergence behavior. For each example we plot the error quantities estimated in Lemma 2.1 and Theorem 2.2. When apropriate we indicate the

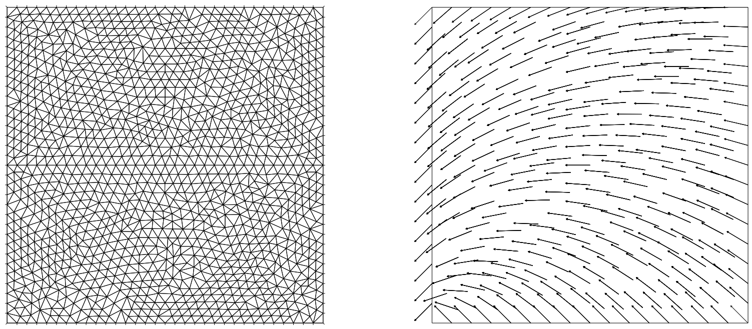

FIG. 5.1. *Left: example of unstructured mesh, $N = 5$. Right: plot of the velocity vector field.*

N	$\|u - u_h\|$	$\|z_h\|$	$\|u_h\|_{S_p} + \|z_h\|_{S_a}$
3	0.038 (–)	0.024	0.57
4	0.012 (1.7)	0.0017	0.24
5	0.0024 (2.3)	0.00043	0.11
6	0.00043 (2.5)	0.00012	0.052
7	0.00010 (2.1)	2.5E-05	0.025
8	2.3E-05 (2.1)	5.3E-06	0.012

TABLE 5.1

Convergence orders of estimated quantities for the Dirichlet problem approximated using piecewise affine elements

experimental convergence order in parenthesis. We report the computational mesh for $N = 5$ and the velocity field β in figure 5. The optimal convergence rate for the stabilizing terms given in Lemma 2.1 is verified in all the numerical examples.

5.1. Dirichlet boundary conditions. In table 5.1 we show the result of the computation when Dirichlet boundary conditions are applied and piecewise affine approximation is used on a sequence of unstructured meshes. We observe that the solution exhibits the preasymptotic convergence rate $h^{\frac{3}{2}}$ under one refinement before achieving the full second order convergence rate in L^2.

In table 5.2 similar data for second order polynomials are presented. Here the asymptotic regime with full convergence is obtained from the first refinement.

5.2. Neumann boundary conditions. We consider the same differential operator, but with (non-homogeneous) Neumann-boundary conditions. This is exactly the problem considered in [11]. The average values of the approximate solutions have been imposed using Lagrange multipliers. In tables 5.3 and 5.4 we observe optimal convergence rates once again as predicted by theory. Observe that in the case of piecewise affine approximation the dual solution z_h comes into the asymptotic regime only on the finer meshes.

5.3. A Cauchy problem. Since we have no complete theory for the ill-posed Cauchy problem we will proceed with a more thorough numerical investigation. First we consider the Cauchy problem obtained by taking $\kappa = 0$ in (4.13). Then we consider a Cauchy problem using the convection–diffusion operator of (4.7) in two different boundary configurations. For all test cases we use the exact solution (5.1) and the

N	$\|u-u_h\|$	$\|z_h\|$	$\|u_h\|_{S_p} + \|z_h\|_{S_a}$
3	0.0014 (–)	0.00041	0.024
4	0.00012 (3.5)	4.6E-05	0.0044
5	8.8E-06 (3.8)	4.6Ee-06	0.00081
6	8.0E-07 (3.5)	6.6E-07	0.00017
7	8.3E-08 (3.3)	8.2E-08	3.7E-05

Table 5.2

Convergence orders of estimated quantities for the Dirichlet problem approximated using piecewise quadratic elements

N	$\|u-u_h\|$	$\|z_h\|$	$\|u_h\|_{S_p} + \|z_h\|_{S_a}$
3	0.028 (–)	0.028 (–)	0.82
4	0.0066 (2.1)	0.016 (0.8)	0.32
5	0.0016 (2.0)	0.0058 (1.5)	0.13
6	0.00039 (2.0)	0.0015 (2.0)	0.060
7	9.7E-05 (2.0)	0.00031 (2.3)	0.028
8	2.3E-05 (2.1)	6.5E-05 (2.3)	0.013

Table 5.3

Convergence orders of estimated quantities for the Neumann problem approximated using piecewise affine elements

stabilisation parameters given above. We present the data for the quantities estimated in Lemma 2.1 and Theorem 2.2, but also the error in the total diffusive flux in the discrete $H^{-1/2}(\partial\Omega)$ norm on the boundary.

$$\|\nabla(u-u_h)\cdot n\|^2_{-\frac{1}{2},h,\partial\Omega} := \int_{\partial\Omega} h(\nabla(u-u_h)\cdot n)^2 \,\mathrm{d}s.$$

5.3.1. Poisson's equation. Here we consider the problem with $\kappa = 0$ in (4.13). We impose the Cauchy data, i.e. both Dirichlet and Neumann data, on boundaries $x=0$, $0<y<1$ and $y=1$, $0<x<1$. In table 5.2 we show the obtained errors when piecewise affine approximation is used and in table 5.2 the results for piecewise quadratic approximation.

First note that in both cases one observes the optimal convergence of the stabilisation terms predicted by Lemma 2.1. For the L^2-norm of the error we observe experimental convergence orders h^α with typically $\alpha \sim 0.25$ for piecewise affine approximation and $\alpha \sim 0.5$ for quadratic approxmation. Higher convergence orders were obtained in both cases for the normal diffusive flux. In figure 5.2, we present a study of the L^2-norm error under variation of the stabilisation parameter. The computations are made on one mesh, with 32 elements per side and the Cauchy problem is solved with $k=1,2$ and different values for $\gamma_{F,1} = \gamma_{F,2}$ with $\gamma_{bc} = 10$ fixed. The level of 10% relative error is indicated by the horizontal dotted line. Observe that the robustness with respect to stabilisation parameters is much better for quadratic approximation. Indeed the 10% error level is met for all parameter values $\gamma_{i,F} \in [2.0E-5, 1]$, whereas in the case of piecewise affine approximation one has to take $\gamma_{1,F} \in [0.003, 0.05]$ approximately. Similar results for the boundary penalty parameter not reported here showed that the method was even more robust under perturbations of γ_{bc}. In the right plot of figure 5.3 we present the contour plot of the error $u-u_h$ and the contour plot of z_h. In both cases the error is concentrated on the boundary without boundary

N	$\|u-u_h\|$	$\|z_h\|$	$\lvert u_h\rvert_{S_p}+\lvert z_h\rvert_{S_a}$
3	0.00061 (–)	0.0020 (–)	0.030
4	6.6E-05 (3.2)	0.00040 (2.3)	0.0054
5	6.5E-06 (3.3)	2.5E-05 (4.0)	0.00099
6	7.1E-07 (3.2)	1.7E-06 (3.9)	0.00020
7	7.9E-08 (3.2)	1.4E-07 (3.6)	4.2E-05

TABLE 5.4

Convergence orders of estimated quantities for the Neumann problem approximated using piecewise quadratic elements

N	$\|u-u_h\|$	$\|z_h\|$	$\lvert u_h\rvert_{S_p}+\lvert z_h\rvert_{S_a}$	$\|\nabla(u-u_h)\cdot n\|_{-\frac{1}{2},h,\partial\Omega}$
3	0.070 (–)	0.59 (–)	2.0 (–)	2.7 (–)
4	0.074 (–)	0.42 (0.49)	0.79 (1.3)	1.3 (1.1)
5	0.037 (1.0)	0.30 (0.49)	0.30 (1.4)	0.75 (0.80)
6	0.029 (0.35)	0.26 (0.2)	0.13 (1.2)	0.51 (0.56)
7	0.024 (0.27)	0.20 (0.37)	0.054 (1.3)	0.33 (0.62)
8	0.020 (0.26)	0.16 (0.32)	0.022 (1.3)	0.21 (0.65)

TABLE 5.5

Convergence orders of estimated quantities for the Poisson Cauchy problem approximated using piecewise affine elements

condition.

5.3.2. The non–coercive convection–diffusion equation. As a last example we consider the Cauchy problem using the non-coercive convection–diffusion operator (3.1). The stability of the problem depends strongly on where the boundary conditions are imposed in relation to the inflow and outflow boundaries. To illustrate this we propose two configurations. Recalling the right plot of figure 5 we observe that the flow enters along the boundaries $y=0$, $y=1$ and $x=1$ and exits on the boundary $x=0$. Note that the strongest inflow takes place on $y=0$ and $x=1$, the flow being close to parallel to the boundary in the right half of the segment $y=1$. We propose the two different Cauchy problem configurations:

Case 1. We impose Dirichlet and Neumann data on the two mixed boundaries $x=0$ and $y=1$.

Case 2. We impose Dirichlet and Neumann data on the two inflow boundaries $y=0$ and $x=1$.

In the first case the outflow portion or the inflow portion of every streamline are include in the Cauchy boundary whereas in the second case the main part of the inflow boundary is included. This highlights two different difficulties for Cauchy problems for the convection–diffusion operator, in case 1 we must solve the problem backward along the characteristics, essentially solving a backward heat equation, whereas in case two the crosswind diffusion must reconstruct missing boundary data.

In tables 5.7 - 5.10, we report the results on the same sequence of unstructured meshes used in the previous examples for piecewise affine and piecewise quadratic approximations and the two problem configurations. First note that in all cases the result of Lemma 2.1 holds as expected. Otherwise the method behaves very differently in the two cases. For case 1 we observe better convergence orders than in the case of the pure Poisson problem, typically $h^{\frac{1}{2}}$ for affine elements and h for quadratic elements in the L^2-norm. Even higher orders are obtained for the global diffusive

N	$\lVert u-u_h \rVert$	$\lVert z_h \rVert$	$\lvert u_h \rvert_{S_p} + \lvert z_h \rvert_{S_a}$	$\lVert \nabla(u-u_h)\cdot n \rVert_{-\frac{1}{2},h,\partial\Omega}$
3	0.031 (–)	0.062 (–)	0.073 (–)	0.92 (–)
4	0.022 (0.49)	0.025 (1.3)	0.014 (2.4)	0.48 (0.94)
5	0.013 (0.76)	0.014 (0.84)	0.0025 (2.5)	0.24 (1.0)
6	0.0088 (0.56)	0.011 (0.35)	0.00047 (2.4)	0.13 (0.88)
7	0.0069 (0.35)	0.0067 (0.72)	8.8E-05 (2.8)	0.080 (0.70)

TABLE 5.6

Convergence orders of estimated quantities for the Poisson Cauchy problem approximated using piecewise quadratic elements, $\gamma = 0.001$, $\gamma_{bc} = 10$

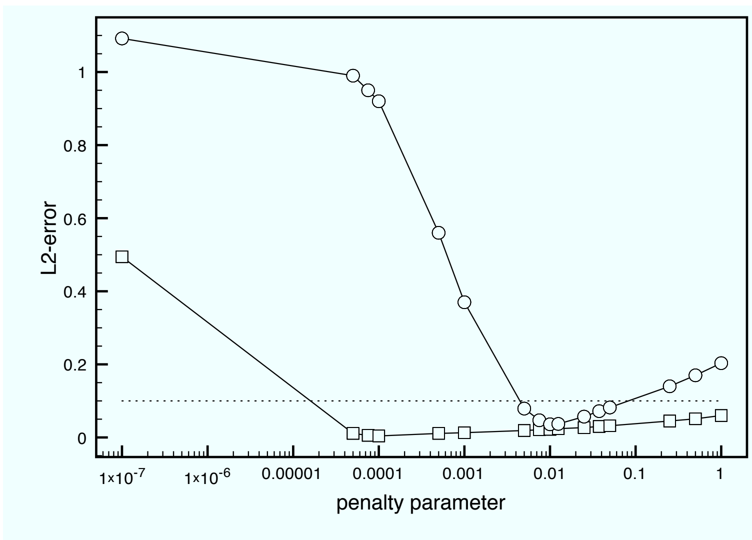

FIG. 5.2. *Study of the L^2-norm error under variation of the stabilisation parameter, circles: affine elements, squares: quadratic elements*

flux in the discrete $H^{-\frac{1}{2}}$ norm. The dual variable z_h on the other hand has very poor convergence, although it is quite small on all meshes in the case of quadratic approximation. Case 2 (control on main part of the inflow) is clearly much more difficult. Convergence orders for both the affine case and the quadratic case are poor (around $\sim h^{\frac{1}{5}}$) and uneven. The diffusive flux still converges approximately as $h^{\frac{1}{2}}$ in both cases. We conclude that the Cauchy convection–diffusion problem is much less ill-posed if for each streamline *either the inflow part or the outflow part lies in the controlled zone*. The fact that we in case 2 controls more of the inflow boundary is unimportant compared to the fact that both the inflow and the outflow are unknown in the boundary portion around the corner $(0,1)$.

6. Concluding remarks. We have proposed a framework for the design of stabilised finite element methods for non-coercive and non-symmetric problems. The fundamental idea is to use an optimisation framework to select the discrete solution on each mesh. This also opens new venues for inverse problems or boundary control problems, where Tichonov regularisation can been introduced in the form of a stabilisation operator with optimal weak consistency properties, eliminating the need to match penalty parameter and mesh size to obtain optimal order. The method

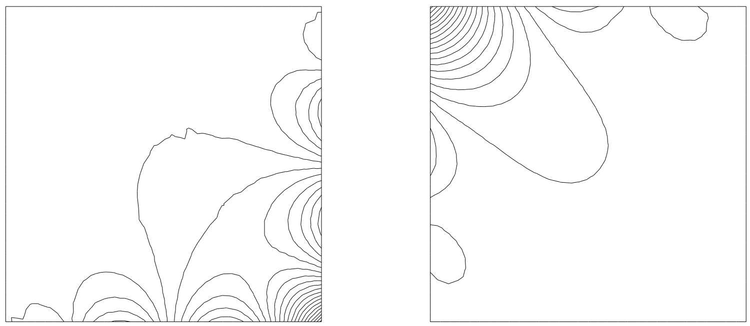

FIG. 5.3. *Contour plots of the error $u - u_h$ (left plot) and the error in the dual variable z_h (right plot).*

N	$\|u-u_h\|$	$\|z_h\|$	$\|u_h\|_{S_p}+\|z_h\|_{S_a}$	$\|\nabla(u-u_h)\cdot n\|_{-\frac{1}{2},h,\partial\Omega}$
3	0.032 (–)	0.044 (–)	1.6 (–)	0.35 (–)
4	0.010 (1.7)	0.020 (1.1)	0.61 (1.4)	0.13 (1.4)
5	0.0045 (1.2)	0.034 (–)	0.24 (1.3)	0.048 (1.4)
6	0.0035 (0.36)	0.052 (–)	0.10 (1.3)	0.018 (1.4)
7	0.0039 (–)	0.056 (–)	0.045 (1.2)	0.0074 (1.3)
8	0.0026 (0.58)	0.059 (–)	0.020 (1.2)	0.0031 (1.3)

TABLE 5.7

Convergence orders of estimated quantities for the convection–diffusion Cauchy problem approximated using piecewise affine elements (case 1)

N	$\|u-u_h\|$	$\|z_h\|$	$\|u_h\|_{S_p}+\|z_h\|_{S_a}$	$\|\nabla(u-u_h)\cdot n\|_{-\frac{1}{2},h,\partial\Omega}$
3	0.13 (–)	0.032 (–)	1.74 (–)	0.44 (–)
4	0.097 (0.42)	0.012 (1.4)	0.63 (1.5)	0.23 (0.94)
5	0.075 (0.37)	0.010 (0.26)	0.24 (1.4)	0.11 (1.1)
6	0.067 (0.16)	0.010 (–)	0.10 (1.3)	0.070 (0.65)
7	0.063 (0.089)	0.0097 (0.044)	0.043 (1.2)	0.047 (0.57)
8	0.056 (0.17)	0.0082 (0.24)	0.018 (1.3)	0.030 (0.65)

TABLE 5.8

Convergence orders of estimated quantities for the convection–diffusion Cauchy problem approximated using piecewise affine elements (case 2)

N	$\|u-u_h\|$	$\|z_h\|$	$\|u_h\|_{S_p}+\|z_h\|_{S_a}$	$\|\nabla(u-u_h)\cdot n\|_{-\frac{1}{2},h,\partial\Omega}$
3	0.0022 (–)	0.0037 (–)	0.096 (–)	0.033 (–)
4	0.00054 (2.0)	0.00089 (2.1)	0.020 (2.3)	0.0091 (1.9)
5	0.00024(1.2)	0.0013 (–)	0.0041 (2.3)	0.0021 (2.1)
6	0.00012 (1.0)	0.00078 (0.74)	0.00096 (2.1)	0.00047 (2.2)
7	5.6E-05 (1.1)	0.00048 (0.70)	0.00022 (2.1)	0.00015 (1.6)

TABLE 5.9

Convergence orders of estimated quantities for the convection–diffusion Cauchy problem approximated using piecewise quadratic elements $\gamma = 0.001$, $\gamma_{bc} = 10$ (case 1)

N	$\|u-u_h\|$	$\|z_h\|$	$\|u_h\|_{S_p}+\|z_h\|_{S_a}$	$\|\nabla(u-u_h)\cdot n\|_{-\frac{1}{2},h,\partial\Omega}$
3	0.020 (–)	0.0014 (–)	0.074 (–)	0.12 (–)
4	0.034 (–)	0.00028 (2.3)	0.013 (2.5)	0.11 (0.12)
5	0.026 (0.39)	0.00011 (1.4)	0.0025 (2.4)	0.065 (0.76)
6	0.024 (0.12)	8.3E-05 (0.4)	0.00046 (2.4)	0.043 (0.60)
7	0.023 (0.06)	3.6E-05 (1.2)	8.7E-05 (2.4)	0.029 (0.57)

TABLE 5.10

Convergence orders of estimated quantities for the convection–diffusion Cauchy problem approximated using piecewise quadratic elements $\gamma = 0.001$, $\gamma_{bc} = 10$ (case 2)

has some other interesting features. In particular for piecewise affine approximation spaces the discrete solution can be shown to exist under very mild assumptions. Both symmetric stabilisation methods and the Galerkin least squares methods are considered in the analysis. Convergence of the method is obtained formally under abstract assumptions on the bilinear form that are shown to hold for three non-trivial examples. The actual performance of the method in practice depends crucially on the stability properties of the underlying PDE and when these are unknown, must be investigated numerically. Sometimes observed convergence orders are unlikely to match those predicted in Theorem 2.2, (except possibly for very small h), due to huge stability constants in the bound (2.3) (cf. the Helmholtz equation for large wave numbers), or poor stability of the dual problem (c.f. the Cauchy problem for Poisson's equation). Another problem that may arise when ill-conditioned problems are considered, is poor conditioning of the system matrix. Even in the case of piecewise affine approximation the stabilisation corresponds to a very weak norm and in case the underlying problem is ill-posed this must be expected to show in the condition number. Clearly preconditioners for the linear systems arising is an important open problem. Other subjects for future work concerns the inclusion of hyperbolic problems in the framework (see [6]) and the application of the method to data assimilation and boundary control.

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